New Results on the Lernmatrix Properties

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Abstract. This paper follows the work presented in [1] and [15]. It shows the advances in the development of a theoretical framework which describes the behavior of the Steinbuch's Lernmatrix, whenever it operates with noisy input patterns. The obtained results allow positioning the Lernmatrix, even four decades after its creation, as a good alternative for pattern classification and recognition.

1 Introduction

The Associative Memories have deserved the attention of numerous international researchers for more than four decades. One of the pioneers was the German scientist Karl Steinbuch who, at the beginning of the Sixties, devised, developed and applied the Lernmatrix [2,3]. The Lernmatrix constitutes a crucial antecedent in the development of the present models of associative memories, and constitutes one on the first successful attempts to codify information into squares, well-known as crossbar [12]. An associative memory has such a fundamental intention: to correctly recover complete patterns from input patterns, which can be altered with additive, subtractive or combined noise. The inherent problem to the operation of the associative memories is normally splited into two clearly distinguishable phases: the learning phase (generation) and the recall phase (operation of the associative memory).

An associative memory **M** can be formulated as an input-output system. The input pattern is represented by a vector column denoted by x, and the output pattern by a vector column denoted by y. Each one of the input patterns forms an association with the corresponding output pattern. An association is denoted by (x, y) and, given a specific positive number k, the corresponding association will be (x^k, y^k).

As a convention, if m and n are the dimensions of the output and input patterns, respectively, it is said that:

$$x^\mu \in A^n \text{ and } y^\mu \in A^m,\ \forall \mu = 1, 2, .., p$$

where A is any set predefined by the associative memory's designer.

The j-th component of a column vector x^μ is denoted by $x_j^\mu \in A^n$.

Analogously, we can represent the j-th component of y^μ by $y_j^\mu \in A^m$.

The associative memory **M** is represented by a matrix whose ij-th component is $m_{ij} \in B$, where B is an appropriate set regarding the values in the set A [11]; the matrix **M** is generated from a finite fundamental set of associations, and its cardinality is denoted by $p \in Z^{+}$. If μ is an index, then the fundamental set is:

$$\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., p\}$$

If $x^{\mu} = y^{\mu} \ \ \forall \mu = 1, 2, .., p$, then the associative memory is called autoassociative; otherwise it's called heteroassociative. For an heteroassociative memory the following statement is true: $\exists \mu \in \{1, 2, .., p\} \ \ x^{\mu} \neq y^{\mu}$.

If a memory **M** responds to a version (that might be altered) of an input pattern with the correct unaltered fundamental output pattern , then the recall is considered perfect. Even though Steinbuch presented the Lernmatrix more than four decades ago, no investigator, including Steinbuch, had studied with scientific rigor the necessary and sufficient conditions for perfect recall of the fundamental set and patterns that do not belong to this one. The former two steps occurred in [1, 15] and this work is another step in the development of a theoretical framework that mathematically describes the behavior of the Lernmatrix, when the input patterns are noisy.

2 The Steinbuch's Lernmatrix

The Lernmatrix is an heteroassociative memory, but it can act as a binary pattern classifier depending on the choice of the output patterns; it is an input-output system that gets a binary input pattern $x^{\mu} \in A^{n}$, $A = \{0,1\}$ and produces the class $y^{\mu} \in A^{m}$ (from m different classes) codified with a simple method: to represent the class $k \in \{1, 2, .., m\}$, you must assign for the output binary pattern y^{μ} the following values: $y_k^{\mu} = 1$, and $y_j^{\mu} = 0$ for $j = 1, 2, .., k-1, k+1, .., m$.

Each component m_{ij} of the Lernmatrix **M** is initialized to zero, and it is updated accordingly to the following rule: $m_{ij} = m_{ij} + \Delta m_{ij}$, where:

$$\Delta m_{ij} = \begin{cases} +\varepsilon & \text{if } y_i^{\mu} = 1 = x_j^{\mu} \\ -\varepsilon & \text{if } y_i^{\mu} = 1 \text{ and } x_j^{\mu} = 0 \\ 0 & \text{otherwise} \end{cases}$$

where ε it's any positive constant previously chosen.

The recalling phase consists of finding the class which an input pattern $x^{\omega} \in A^n$ belongs. Finding the class means to get the components of the vector $y^{\omega} \in A^m$ which corresponds to x^{ω}; accordingly to the constructing method of all y^{μ}, the class should be obtained without ambiguity.

The i-th coordinate y_i^{ω} of the class vector $y^{\omega} \in A^m$ is obtained according to the next expression, where $\bigcup$ is the maximum operator:

$$y_i^{\omega} = \begin{cases} 1 \text{ if } \sum_{j=1}^{n} m_{ij} x_j^{\omega} = \bigcup_{h=1}^{m} \left[\sum_{j=1}^{n} m_{hj} x_j^{\omega} \right] \\ 0 \text{ otherwise} \end{cases}$$

3 A Healthy Change of Notation

The way Steinbuch proposed the learning and the recalling phases is not adequate for analyze the memory properties. That's why we proposed an alternative characterization of both phases, in which the concept of Steinbuch's function is presented [1].

Definition 1. *A Steinbuch's function is any function $f : \mathbf{R} \to \mathbf{R}$ with the property:*

$$f(0) = -1; f(1) = 1$$

Definition 2. *Let $f : \mathbf{R} \to \mathbf{R}$ be a Steinbuch's function. Al Steinbuch's vectorial function for f is any function $f : \mathbf{R} \to \mathbf{R}$ with the property:*

$$F(x) = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \cdots \\ f(x_n) \end{pmatrix}$$

Now, it's possible to propose an alternative and better (for this paper's goals) characterization for the learning and recalling steps.

Learning phase for the Lernmatrix

Let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ be a fundamental set and F an *Steinbuch's vectorial function for* f. The Lernmatrix M for the fundamental set is built accordingly to the next rule:

$$M = \sum_{\mu=1}^{m} y^{\mu} \bullet (F(x^{\mu}))^{T}$$

Recalling phase for the Lernmatrix

Let M be a Lernmatrix and x^{ω} a n-dimensional pattern. The pattern $\tilde{y}^{\omega}$ obtained from x^{ω} and M is determinate as follows:

$$z^{\omega} = M \bullet x^{\omega}$$

$$y_i^{\omega} = \begin{cases} 1 \text{ if } z_i^{\omega} = \bigcup_{h=1}^{m} z_h^{\omega} \\ 0 \text{ otherwise} \end{cases}$$

where $\tilde{y}^{\omega}$ is not necessarily equal to y^{ω}. Indeed, if $\tilde{y}^{\omega} = y^{\omega}$, then the recalling is perfect.

4 Learning and Perfect Recall Conditions for non-Noisy Patterns

In [1, 15] the authors presented several novel results concerning the necessary and sufficient conditions for perfect recall for the Lernmatrix. In this paper, as will be shown in the next section, we provide new results that regard with the learning and recall for noisy patterns. For the former results we can recall the following:

Definition 3. *Let* $A = \{0,1\}$ *and let* $x^{\omega} \in A^{n}$ *a pattern. We call characteristic set of* x^{ω} *to the index set* $T^{\omega} = \{i \mid x_i^{\omega} = 1\}$*. Its cardinality is denoted* $|T^{\omega}|$

This concept is the spine of this framework. The first step will be to establish a relationship between this concept, the order relations and the Lernmatrix.

Lemma 1. Let $A = \{0,1\}$ and let x^{α}, $x^{\beta} \in A^{n}$ be two patterns, then $x^{\alpha} \le x^{\beta} \Leftrightarrow T^{\alpha} \subseteq T^{\beta}$

The proof of this lemma appears in [15]. The lemma means that an order relation between patterns implies an order relation between their characteristic sets and vice versa.

Furthermore, we showed in [15] the relation between the recalling phase and the characteristic sets:

Lemma 2. Let M a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1,2,..,m\}$ be the fundamental set of M, x^{ω} a random pattern and $z^{\omega} = M \bullet x^{\omega}$. Then, $z_k^{\omega} = \varepsilon(2|T^{k} \cap T^{\omega}| - |T^{\omega}|)$.

The next step is to establish a connection between the characteristic sets and the necessary and sufficient conditions for perfect recalling. In the next 2 lemmas, whose proofs are also given in [15], we synthesized the functionality of the Lernmatrix: these results opened the door to understand its most important properties.

Lemma 3. Let M be a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^{ω} a pattern, which may belong or not to the fundamental set of M. Then, the recovered pattern from x^{ω} and M will be $\tilde{y}^{\alpha}$, with $y^{\alpha} \leq \tilde{y}^{\alpha}$, $\alpha \in \{1, 2, .., m\}$ if and only if $|T^{\alpha} \cap T^{\omega}| \geq |T^{\beta} \cap T^{\omega}| \ \forall \beta \in \{1, 2, .., m\}$ and $\alpha \neq \beta$.

Lemma 4. Let M be a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^{ω} a pattern, which may belong or not to the fundamental set of M. Then, the recovered pattern from x^{ω} and M will be $\tilde{y}^{\alpha}$, with $y^{\alpha} = \tilde{y}^{\alpha}$, $\alpha \in \{1, 2, .., m\}$ if and only if $|T^{\alpha} \cap T^{\omega}| > |T^{\beta} \cap T^{\omega}| \ \forall \beta \in \{1, 2, .., m\}$ and $\alpha \neq \beta$.

Now, we also present a new lemma, which is equivalent to lemma 4, but it'll be useful when the input patterns be altered with additive and mixed noise.

Lemma 5. Let M a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and x^{ω} a pattern, which can belong or not to the fundamental set of M. Then, the recovered pattern from x^{ω} and M will be $\tilde{y}^{\alpha}$, with $y^{\alpha} = \tilde{y}^{\alpha}$, $\alpha \in \{1, 2, .., m\}$ if and only if $|(T^{\alpha} - T^{\beta}) \cap T^{\omega}| > |(T^{\omega} - T^{\alpha}) \cap T^{\beta}| \ \forall \beta \in \{1, 2, .., m\}$ and $\alpha \neq \beta$.

Proof. Let M a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and x^{ω} an n-dimentional pattern. Let α an arbitraty index such that $\alpha \in \{1, 2, .., m\}$ and let $\tilde{y}^{\alpha}$, with $y^{\alpha} = \tilde{y}^{\alpha}$ the recovered pattern for x^{ω}.

According to lemma 4,

$$|T^{\alpha} \cap T^{\omega}| > |T^{\beta} \cap T^{\omega}| \ \forall \beta \in \{1, 2, .., m\} \text{ and } \alpha \neq \beta .$$

But, we can rewrite the last inequality as follows:

$$|T^{\alpha} \cap T^{\omega} \cap \square| > |T^{\beta} \cap T^{\omega} \cap \square|$$

$$|T^{\alpha} \cap T^{\omega} \cap \left[T^{\beta} \cup (T^{\beta})^{C}\right]| > |T^{\beta} \cap T^{\omega} \cap \left[T^{\alpha} \cup (T^{\alpha})^{C}\right]|$$

And by the distributive law:

$$\begin{array}{l} |\left(T^{\alpha} \cap T^{\omega} \cap T^{\beta}\right) \cup \left[T^{\alpha} \cap T^{\omega} \cap (T^{\beta})^{C}\right]| > \\ |\left(T^{\beta} \cap T^{\omega} \cap T^{\alpha}\right) \cup \left[T^{\beta} \cap T^{\omega} \cap (T^{\alpha})^{C}\right]| \end{array}.$$

Then, the sets $\left(T^{\alpha} \cap T^{\omega} \cap T^{\beta}\right)$ and $\left[T^{\alpha} \cap T^{\omega} \cap (T^{\beta})^{C}\right]$ are disjoint sets; and the sets $\left(T^{\beta} \cap T^{\omega} \cap T^{\alpha}\right)$ and $\left[T^{\beta} \cap T^{\omega} \cap (T^{\alpha})^{C}\right]$ are also disjoint, so:

$$\begin{array}{l} \left|\left(T^{\alpha} \cap T^{\omega} \cap T^{\beta}\right)\right| + \left|\left[T^{\alpha} \cap T^{\omega} \cap (T^{\beta})^{C}\right]\right| > \\ \left|\left(T^{\beta} \cap T^{\omega} \cap T^{\alpha}\right)\right| + \left|\left[T^{\beta} \cap T^{\omega} \cap (T^{\alpha})^{C}\right]\right| \end{array}.$$

Then,

$$\left|\left[T^{\alpha} \cap T^{\omega} \cap (T^{\beta})^{C}\right]\right| > \left|\left[T^{\beta} \cap T^{\omega} \cap (T^{\alpha})^{C}\right]\right|.$$

And by the definition of substraction between sets:

$$|\left(T^{\alpha} - T^{\beta}\right) \cap T^{\omega}| > |\left(T^{\omega} - T^{\alpha}\right) \cap T^{\beta}|$$

So we can conclude that:

$$|\left(T^{\alpha} - T^{\beta}\right) \cap T^{\omega}| > |\left(T^{\omega} - T^{\alpha}\right) \cap T^{\beta}| \; \forall \beta \in \{1, 2, .., m\} \text{and } \alpha \neq \beta\,.$$

Now, we prove two theorems that are involved in the perfect recall for the Lernmatrix.

Theorem 1. Let M be a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^{α} a pattern in the fundamental set of M. Then, the recalled pattern with x^{α} and M is $\tilde{y}^{\alpha}$, and $y^{\alpha} \leq \tilde{y}^{\alpha}$,

Proof. Let M be a Lernmatrix, and let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^{α} a pattern in the fundamental set of M. Let $\tilde{y}^{\alpha}$ the recalled pattern with x^α and M. If α and β are arbitrary indexes such that $\alpha, \beta \in \{1, 2, .., m\}$ and α≠β, according with lemma 3 we have:

$$y^\alpha \leq \tilde{y}^\alpha \Leftrightarrow |T^\alpha \cap T^\alpha| \geq |T^\beta \cap T^\alpha|$$

$$y^\alpha \leq \tilde{y}^\alpha \Leftrightarrow |T^\alpha| \geq |T^\beta \cap T^\alpha|$$

Then, for any two sets A and B , $A \supseteq B \Leftrightarrow |A| \geq |B|$ and also $T^\alpha \supseteq T^\beta \cap T^\alpha$, then, the proposition $|T^\alpha| \geq |T^\beta \cap T^\alpha|$ is true, so $y^\alpha \leq \tilde{y}^\alpha$ is true. Finally, using the fact that α and β were chosen arbitrarily the theorem is proved.

Theorem 2. Let M be a Lernmatrix, let $\{(x^\mu, y^\mu) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^α a pattern in the fundamental set of M.. Then, the recalled pattern with x^α and M is $\tilde{y}^\alpha$, with $y^\alpha = \tilde{y}^\alpha$, if and only if the proposition $\forall \beta \in \{1, 2, .., m\}, \alpha \neq \beta$ and $\neg(x^\alpha \leq x^\beta)$ is true.

Proof. Let M be a Lernmatrix, and let $\{(x^\mu, y^\mu) \mid \mu = 1, 2, .., m\}$ be the fundamental set of M and x^α a pattern in the fundamental set of M. Let $\tilde{y}^\alpha$ the recalled pattern with x^α and M. If α and β are arbitrary indexes such that $\alpha, \beta \in \{1, 2, .., m\}$ and α≠β, according with lemma 4 we have:

$$y^\alpha = \tilde{y}^\alpha \Leftrightarrow |T^\alpha \cap T^\alpha| > |T^\beta \cap T^\alpha|$$

$$y^\alpha = \tilde{y}^\alpha \Leftrightarrow |T^\alpha| > |T^\beta \cap T^\alpha|$$

Then, for any two sets A and B , $A \supset B \Rightarrow |A| > |B|$ and $T^\alpha \supset T^\beta \cap T^\alpha$ if and only if $\neg(T^\alpha \subseteq T^\beta)$, then:

$$|T^\alpha| > |T^\beta \cap T^\alpha| \Leftrightarrow \neg(T^\alpha \subseteq T^\beta)$$

According to lemma 1, $x^\alpha \leq x^\beta \Leftrightarrow T^\alpha \subseteq T^\beta$ so,

$$|T^\alpha| > |T^\beta \cap T^\alpha| \Leftrightarrow \neg(x^\alpha \leq x^\beta)$$

Then, by transitivity, $y^\alpha = \tilde{y}^\alpha \Leftrightarrow \neg(x^\alpha \leq x^\beta)$. Finally, using the fact that α and β were chosen arbitrarily the theorem is proved.

5 New Results on Noisy Input Patterns

5.1 Sustractive noise

In the following 2 theorems, we establish the sufficient conditions for perfect recall when the input pattern is a noisy pattern, in particular when the pattern is affected by sustractive noise:

Theorem 3. Let M be a Lernmatrix, let $\{(x^\mu, y^\mu) \mid \mu = 1,2,..,m\}$ be the fundamental set of M and $\tilde{x}^\alpha$ a noisy pattern such that $\tilde{x}^\alpha \leq x^\alpha$. Then, the recalled pattern is $\tilde{y}^\alpha$, and $y^\alpha \leq \tilde{y}^\alpha$,

Proof. Let M be a Lernmatrix, and let $\{(x^\mu, y^\mu) \mid \mu = 1,2,..,m\}$ be the fundamental set of M and $\tilde{x}^\alpha$ a noisy pattern such that $\tilde{x}^\alpha \leq x^\alpha$. Let $\tilde{y}^\alpha$ be the recalled pattern with x^α and M. According with lemma 3 and let α and β arbitrary indexes such that $\alpha, \beta \in \{1,2,..,m\}$ and α≠β.

$$y^\alpha \leq \tilde{y}^\alpha \Leftrightarrow |T^\alpha \cap \tilde{T}^\alpha| \geq |T^\beta \cap \tilde{T}^\alpha|$$

$$y^\alpha \leq \tilde{y}^\alpha \Leftrightarrow |\tilde{T}^\alpha| \geq |T^\beta \cap \tilde{T}^\alpha|$$

Then, for any two sets A and B , $A \supseteq B \Leftrightarrow |A| \geq |B|$ and also $\tilde{T}^\alpha \supseteq T^\beta \cap \tilde{T}^\alpha$, then, the proposition $|\tilde{T}^\alpha| \geq |T^\beta \cap \tilde{T}^\alpha|$ its true, so $y^\alpha \leq \tilde{y}^\alpha$ is true. Finally, in the fact that α and β were chosen arbitrarily the theorem is proved.

Theorem 4. Let M be a Lernmatrix, let $\{(x^\mu, y^\mu) \mid \mu = 1,2,..,m\}$ be the fundamental set of M and $\tilde{x}^\alpha$ a noisy pattern such that $\tilde{x}^\alpha \leq x^\alpha$. Then, the recalled pattern with $\tilde{x}^\alpha$ and M is $\tilde{y}^\alpha$, with $y^\alpha = \tilde{y}^\alpha$, if and only if the proposition $\forall \beta \in \{1,2,..,m\}, \alpha \neq \beta$ and $\neg(\tilde{x}^\alpha \leq x^\beta)$ is true.

Proof. Let M be a Lernmatrix, and $\{(x^\mu, y^\mu) \mid \mu = 1,2,..,m\}$ the fundamental set of M and $\tilde{x}^\alpha$ a noisy pattern such that $\tilde{x}^\alpha \leq x^\alpha$. Let $\tilde{y}^\alpha$ be the recalled pattern with $\tilde{x}^\alpha$ and M. According with lemma 4 and let α and β arbitrary indexes such that $\alpha, \beta \in \{1,2,..,m\}$ and α≠β.

$$y^\alpha = \tilde{y}^\alpha \Leftrightarrow |T^\alpha \cap \tilde{T}^\alpha| > |T^\beta \cap \tilde{T}^\alpha|$$

$$y^\alpha = \tilde{y}^\alpha \Leftrightarrow |\tilde{T}^\alpha| > |T^\beta \cap \tilde{T}^\alpha|$$

Then, for any two sets A and B, $A \supset B \Rightarrow |A| > |B|$ and $\tilde{T}^{\alpha} \supset T^{\beta} \cap \tilde{T}^{\alpha}$ if and only if $\neg(\tilde{T}^{\alpha} \subseteq T^{\beta})$, then:

$$|\tilde{T}^{\alpha}| > |T^{\beta} \cap \tilde{T}^{\alpha}| \Leftrightarrow \neg(\tilde{T}^{\alpha} \subseteq T^{\beta})$$

According to lemma 1 $\tilde{x}^{\alpha} \leq x^{\beta} \Leftrightarrow \tilde{T}^{\alpha} \subseteq T^{\beta}$, so

$$|\tilde{T}^{\alpha}| > |T^{\beta} \cap \tilde{T}^{\alpha}| \Leftrightarrow \neg(\tilde{x}^{\alpha} \leq x^{\beta})$$

Then, by transitivity, $y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow \neg(\tilde{x}^{\alpha} \leq x^{\beta})$. Finally, using the fact that α and β were chosen arbitrarily the theorem is proved.

5.2 Additive noise

Finally, we prove that Lermatrix may recall patterns affected with additive noise, but in this case, we can find an upper bound for the amount of noise supported by the associative memory.

Theorem 5. Let M a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and $\tilde{x}^{\alpha}$ a noisy pattern such that $x^{\alpha} \leq \tilde{x}^{\alpha}$. Then, the recalled pattern with $\tilde{x}^{\alpha}$ and M is $\tilde{y}^{\alpha}$, with $y^{\alpha} = \tilde{y}^{\alpha}$, if and only if the proposition $|(T^{\alpha} - T^{\beta})| > |(\tilde{T}^{\alpha} - T^{\alpha}) \cap T^{\beta}| \; \forall \beta \in \{1, 2, .., m\}, \alpha \neq \beta$ is true.

Proof. Let M a Lernmatrix, $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and $\tilde{x}^{\alpha}$ a noisy pattern such that $x^{\alpha} \leq \tilde{x}^{\alpha}$. Let $\tilde{y}^{\alpha}$ the recalled pattern with $\tilde{x}^{\alpha}$ and M. According with lemma 5 and let α and β arbitrary indexes such that $\alpha, \beta \in \{1, 2, .., m\}$ and α≠β.

$$y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow |(T^{\alpha} - T^{\beta}) \cap \tilde{T}^{\alpha}| > |(\tilde{T}^{\alpha} - T^{\alpha}) \cap T^{\beta}|$$

$$y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow \left|\left[T^{\alpha} \cap \tilde{T}^{\alpha} \cap (T^{\beta})^{C}\right]\right| > |(\tilde{T}^{\alpha} - T^{\alpha}) \cap T^{\beta}|$$

Then, according to lemma 1, $T^{\alpha} \cap \tilde{T}^{\alpha} = T^{\alpha}$, so:

$$y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow \left|\left[T^{\alpha} \cap (T^{\beta})^{C}\right]\right| > |(\tilde{T}^{\alpha} - T^{\alpha}) \cap T^{\beta}|$$

and

$$y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow \left|\left[T^{\alpha} - T^{\beta}\right]\right| > \left|\left(\tilde{T}^{\alpha} - T^{\alpha}\right) \cap T^{\beta}\right|$$

Finally, in the fact that α and β were chosen arbitrarily the theorem is proved.

We can notice, that $\left|\left[T^{\alpha} - T^{\beta}\right]\right|$ is a constant term once the Lernmatrix is generated. Even more, $\left(\tilde{T}^{\alpha} - T^{\alpha}\right)$ is the additive noise in the pattern, so $\left|\left[T^{\alpha} - T^{\beta}\right]\right|$ is an upper bound for the amount of additive noise for a pattern x^{α}.

5.3 Mixed noise

As additive noise, in this section we can find an upper bound for the amount of mixed noise supported by the associative memory.

Theorem 6. Let M a Lernmatrix, let $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and $\tilde{x}^{\alpha}$ a pattern affected with mixed noise. Then, the recalled pattern with $\tilde{x}^{\alpha}$ and M is $\tilde{y}^{\alpha}$, with $y^{\alpha} = \tilde{y}^{\alpha}$, if and only if the proposition $\left|\left(T^{\alpha} - T^{\beta}\right) \cap \tilde{T}^{\alpha}\right| > \left|\left(\tilde{T}^{\alpha} - T^{\alpha}\right) \cap T^{\beta}\right| \; \forall \beta \in \{1, 2, .., m\}, \alpha \neq \beta$ is true.

Proof. Let M a Lernmatrix, $\{(x^{\mu}, y^{\mu}) \mid \mu = 1, 2, .., m\}$ the fundamental set of M and $\tilde{x}^{\alpha}$ a pattern affected with mixed noise. Let $\tilde{y}^{\alpha}$ the recalled pattern with $\tilde{x}^{\alpha}$ and M. According with lemma 5 and let α and β arbitrary indexes such that $\alpha, \beta \in \{1, 2, .., m\}$ and α≠β.

$$y^{\alpha} = \tilde{y}^{\alpha} \Leftrightarrow \left|\left(T^{\alpha} - T^{\beta}\right) \cap \tilde{T}^{\alpha}\right| > \left|\left(\tilde{T}^{\alpha} - T^{\alpha}\right) \cap T^{\beta}\right|$$

Finally, in the fact that α and β were chosen arbitrarily the theorem is proved.

Since $\left(T^{\alpha} - T^{\beta}\right) \supseteq \left(T^{\alpha} - T^{\beta}\right) \cap \tilde{T}^{\alpha}$, then $\left|\left(T^{\alpha} - T^{\beta}\right)\right| \geq \left|\left(T^{\alpha} - T^{\beta}\right) \cap \tilde{T}^{\alpha}\right|$, so we can rewritte the result of theorem 6 as follows:

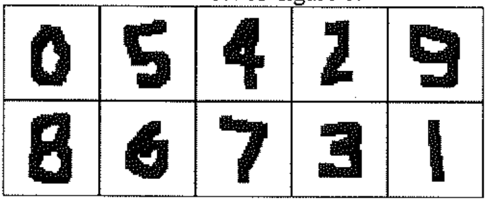
$$\left|\left(T^{\alpha} - T^{\beta}\right)\right| \geq \left|\left(T^{\alpha} - T^{\beta}\right) \cap \tilde{T}^{\alpha}\right| > \left|\left(\tilde{T}^{\alpha} - T^{\alpha}\right) \cap T^{\beta}\right|$$

We can notice, that $\left|\left[T^{\alpha} - T^{\beta}\right]\right|$ is a constant term once the Lernmatrix is generated; $\left(\tilde{T}^{\alpha} - T^{\alpha}\right)$ is the additive noise in the pattern, so $\left|\left[T^{\alpha} - T^{\beta}\right]\right|$ is an upper.

bound for the amount of additive noise for a pattern x^α, but since $\left|\left(T^\alpha - T^\beta\right)\right| \geq \left|\left(T^\alpha - T^\beta\right) \cap \tilde{T}^\alpha\right|$, the Lernmatix is less robust to mixed noise than additive noise.

6 Experimental Results

In this section, we show the experimental support to the theoretical results. We generated a Lernmatrix with the fundamental set of figure 1.



The input patterns were images with 50x50 pixels, and the output patterns were buildt according to the method described in section 2. Then, we generate 100 noisy patterns for each fundamental pattern, for each kind of noise and for each and for each amount of noise. This fundamental has the property $\forall \beta \in \{1, 2, .., m\}, \alpha \neq \beta$ and $\neg(x^\alpha \leq x^\beta)$ is true, so the memory is perfect according to theorem 2.

For input patterns with sustractive noise, we obtained perfect recalling form 0 to 87% of sustractive noise and other results are presented in table 1. Oterwise, the recovered pattern is greater than the expected pattern, according to theorem 3.

Noise	88	90	95	98	99
% of pertect recalling	99.8	99.6	96.1	76.6	42.9

For input patterns with additive noise, we obtained perfect recalling form 0 to 69% of additive noise and other results are presented in the next table.

Noise	70	100	150	200	220
% of pertect recalling	99.8	95.2	59.5	20	18.9

For input patterns with mixed noise, we obtained perfect recalling form 0 to 5% of mixed noise and other results are presented in the next table:

Noise	6	10	20	30	50
% of pertect recalling	99.9	94.3	88.8	75.7	25.3

7 Conclusions and Future Work

This work is another step in the development of a theoretical framework that mathematically describes the behavior of the Lernmatrix. The relevance of the results presented in this article allows us to affirm that this work lays clear ways for those who are interested in making future research on the subject, this is: to search for some other properties that may exhibit this associative memory; the conditions under which saturation occurs when several patterns that comprise of a same class are used; the possible creation of a new version of the Lernmatrix, that not only works on binary patterns. It is suggested to combine all this baggage of ideas related to the Lernmatrix from Steinbuch, with known models of associative memories that enjoys of great prestige, such as: The model of Kanerva, the Hopfield memory, the BAM of Kosko, the Morphological Associative Memories and the $\alpha\beta$ Associative Memories.

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